Scattering Amplitudes from Multivariate Polynomial Division

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Abstract

We show that the evaluation of scattering amplitudes can be formulated as a problem of multivariate polynomial division, with the components of the integration-momenta as indeterminates. We present a recurrence relation which, independently of the number of loops, leads to the multi-particle pole decomposition of the integrands of the scattering amplitudes. The recursive algorithm is based on the Weak Nullstellensatz Theorem and on the division modulo the Gröbner basis associated to all possible multi-particle cuts. We apply it to dimensionally regulated one-loop amplitudes, recovering the well-known integrand-decomposition formula. Finally, we focus on the maximum-cut, defined as a system of on-shell conditions constraining the components of all the integration-momenta. By means of the Finiteness Theorem and of the Shape Lemma, we prove that the residue at the maximum-cut is parametrized by a number of coefficients equal to the number of solutions of the cut itself.

Keywords: Scattering amplitudes, Unitarity, Polynomial Division

1. Introduction

Scattering amplitudes in quantum field theories are analytic functions of the momenta of the particles involved in the scattering process, and can be determined by their singularity structure. The multi-particle factorization properties of the amplitudes are exposed when propagating particles go on their mass-shell [1–4].

The investigation of the residues at the singular points has been fundamental for discovering new relations fulfilled by scattering amplitudes. The BCFW recurrence relation [4], its link to the leading singularity of one-loop amplitudes [2], and the OPP integrand-decomposition formula for one-loop integrals [5] have shown the underlying simplicity beneath the rich mathematical structure of quantum field theory. Moreover they have become efficient techniques leading to quantitative predictions at the next-to-leading order in perturbation theory [6–13].

The integrand reduction methods [5] allow to de-

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compose one-loop amplitudes in terms of Master Integrals (MI's) without performing the loop integration, and are based on the multi-particle pole expansion of the integrand. The expansion is equivalent to the decomposition of the numerator in terms of (a combination of) products of denominators, with polynomial coefficients. In the context of an integrand-reduction, any integration is replaced by *polynomial fitting*.

The first extension of the *integrand reduction method* beyond one-loop was proposed in [14], and it was used to reproduce the results of two-loop 5-point planar and non-planar amplitudes in $\mathcal{N} = 4$ SYM [15, 16]. A key point of the higher-loop extension is the proper parametrization of the residues at the multi-particle poles. Each residue is a multivariate polynomial in the *irreducible scalar products* (ISP's) among the loop momenta and either external momenta or polarization vectors constructed out of them. ISP's cannot be expressed in terms of denominators, thus any monomial formed by ISP's is the numerator of a potential MI which may appear in the final result. Hence, a systematic classification of the polynomial structures of the residues is mandatory. In [14], the residues have been obtained by relating the ISPs to monomials in the components of the loop momenta expressed in a basis chosen according to the topology of the on-shell diagram.

Badger, Frellesvig and Zhang [17] combined on-shell conditions with Gram-identities [18] to limit the number of monomials appearing in the residues. This technique was applied to the integrand decomposition of two-loop 4-point planar and non-planar diagrams in supersymmetric as well as non-supersymmetric YM theories.

In this work, we show that the shape of the residues is uniquely determined by the on-shell conditions alone, without any additional constraint. We derive a simple *integrand recurrence relation* that generates the required multi-particle pole decomposition for arbitrary amplitudes, independently of the number of loops.

The algorithm treats the numerator and the denominators of any Feynman integrand, as multivariate polynomials in the components of the loop variables. The properties of multivariate polynomials have been extensively studied in the mathematical literature, see *e.g.* [19–25]. The method uses both the *weak Nullstellensatz theorem* and the *multivariate polynomial division* modulo appropriate Gröbner basis [19]. In the context of the integrand reduction, univariate polynomial division has been already introduced in [26] to improve the decomposition of one-loop scattering amplitudes.

The algorithm, which is described in Section II, relies on general properties of the loop integrand:

- When the number n of denominators is larger than the total number of the components of the loop momenta, the *weak Nullstellensatz theorem* yields the trivial reduction of an n-denominator integrand in terms of integrands with $(n-1)$ denominators.

- When n is equal or less than the total number of components of the loop momenta, we divide the numerator modulo the Gröbner basis of the n-ple cut, namely modulo a set of polynomials vanishing on the same on-shell solutions as the cut denominators. The *remainder* of the division is the *residue* of the n-ple cut. The *quotients* generate integrands with $(n-1)$ denominators which should undergo the same decomposition.

- By iterating this procedure, we extract the polynomial forms of *all* residues. The algorithm will stop when all cuts are exhausted, and no denominator is left, leaving us with the integrand reduction formula.

In Section III we apply the algorithm to a generic one-loop integrand, reproducing the d-dimensional integrand decomposition formula [5, 27–29].

In Section IV we conclude by proving a theorem on the *maximum-cuts*, i.e. the cuts defined by the maximum number of on-shell conditions which can be simultaneously satisfied by the loop momenta. The on-shell conditions of a maximum cut lead to a zero-dimensional system. The *Finiteness Theorem* and the *Shape Lemma* ensure that the residue at the maximum-cut is parametrized by n_s coefficients, where n_s is the number of solutions of the multiple cut-conditions. This guarantees that the corresponding residue can always be reconstructed by evaluating the numerator at the solutions of the cut.

During the completion of this work, Zhang has presented an algorithm [30] embedding the ideas presented in [17] within more general techniques of algebraic geometry, among which the division modulo Gröbner basis is used as well.

2. Multivariate polynomial division

In what follows, we assume 4-dimensional loop-momenta. Extensions to higher-dimensional cases, according to the chosen dimensional regularization scheme, can be treated analogously - as we will show when discussing the one-loop integrand reduction.

The integrand reduction methods [5, 14, 17, 26–29, 31–34] recast the problem of computing ℓ-loop amplitudes with n denominators as the reconstruction of integrand functions of the type

$$\mathcal{I}_{i_1\cdots i_n} \equiv \frac{\mathcal{N}_{i_1\cdots i_n}(q_1,\ldots,q_\ell)}{D_{i_1}(q_1,\ldots,q_\ell)\cdots D_{i_n}(q_1,\ldots,q_\ell)} \, , \quad (1)$$

where $q_1,\ldots,q_\ell$ are integration momenta. The generic propagator can be written as follows:

$$D_i = \left(\sum_{j=1}^{\ell} \alpha_j \, q_j + p_i\right)^2 - m_i^2 \, , \; \alpha_j \in \{0,\pm 1\} \, . \quad (2)$$

The numerator $\mathcal{N}_{i_1\cdots i_n}$ and any of the denominators D_i are polynomial in the components of the loop momenta, say $\mathbf{z} \equiv (z_1,\ldots,z_{4\ell})$, *i.e.*

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}(\mathbf{z})}{D_{i_1}(\mathbf{z})\cdots D_{i_n}(\mathbf{z})} \, . \quad (3)$$

Let us consider the ideal generated by the n denominators in Eq. (3) ,

$$\begin{aligned} \mathcal{J}_{i_1\cdots i_n} &= \langle D_{i_1},\cdots,D_{i_n}\rangle \\ &\equiv \left\{\sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}]\right\}, \end{aligned}$$

where $P[\mathbf{z}]$ is the set of polynomials in $\mathbf{z}$. The common zeros of the elements of $\mathcal{J}_{i_1\cdots i_n}$ are exactly the common zeros of the denominators.

The multi-pole decomposition of Eq. (1) is explicitly achieved by performing multivariate polynomial division, yielding an expression of $\mathcal{N}_{i_1\cdots i_n}$ in terms of denominators and residues.
We construct a Gröbner basis [19] (see Ch. 2 of [20]), generating the ideal $\mathcal{J}_{i_1\cdots i_n}$ with respect to a chosen monomial order,

$$\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}),\ldots,g_m(\mathbf{z})\} \, . \quad (4)$$

Unless otherwise indicated, we will assume lexicographic order.

In this formalism, the n-ple cut-conditions $D_{i_1} = \ldots = D_{i_n} = 0$, are equivalent to $g_1 = \ldots = g_m = 0$. The number m of elements of the Gröbner basis is the *cardinality* of the basis. In general, m is different from n. We then consider the multivariate division of $\mathcal{N}_{i_1\cdots i_n}$ modulo $\mathcal{G}_{i_1\cdots i_n}$ (see Ch. 2, Thm. 3 of [20]),

$$\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \Gamma_{i_1\cdots i_n} + \Delta_{i_1\cdots i_n}(\mathbf{z}) \, , \quad (5)$$

where $\Gamma_{i_1\cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z})$ is a compact notation for the sum of the products of the quotients $\mathcal{Q}_i$ and the divisors g_i. The polynomial $\Delta_{i_1\cdots i_n}$ is the remainder of the division. Since $\mathcal{G}_{i_1\cdots i_n}$ is a Gröbner basis, the remainder is uniquely determined once the monomial order is fixed.
The term $\Gamma_{i_1\cdots i_n}$ belongs to the ideal $\mathcal{J}_{i_1\cdots i_n}$, thus it can be expressed in terms of denominators, as

$$\Gamma_{i_1\cdots i_n} = \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \, . \quad (6)$$

The explicit form of $\mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}$ can be found by expressing the elements of the Gröbner basis in terms of the denominators.

2.1. Reducibility criterion.

An integrand $\mathcal{I}_{i_1\cdots i_n}$ is said to be reducible if it can be written in terms of lower-point integrands: that happens when the numerator can be written in terms of denominators. The concept of *reducibility* can be formalized in algebraic geometry. Indeed a direct consequence of Eqs. (5) and (6) is the following

Proposition 2.1. *The integrand $\mathcal{I}_{i_1\cdots i_n}$ is reducible iff the remainder of the division modulo a Gröbner basis vanishes, i.e. iff $\mathcal{N}_{i_1\cdots i_n} \in \mathcal{J}_{i_1\cdots i_n}$.*

Proposition 2.1 allows to prove

Proposition 2.2. *An integrand $\mathcal{I}_{i_1,\ldots,i_n}$ is reducible if the cut $(i_1,\ldots,i_n)$ leads to a system of equations with no solution.*

Proof. In this case, the system is over-constrained. The n propagators cannot vanish simultaneously, i.e.

$$D_{i_1}(\mathbf{z}) = \cdots = D_{i_n}(\mathbf{z}) = 0 \quad (7)$$

has no solution. Therefore, according to the *weak Nullstellensatz* theorem (Thm. 1, Ch. 4 of [20]),

$$1 = \sum_{\kappa=1}^{n} w_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \;\in \mathcal{J}_{i_1\cdots i_n} \, , \quad (8)$$

for some $\omega_\kappa \in P[\mathbf{z}]$. Irrespective of the monomial order, a (reduced) Gröbner basis is $\mathcal{G} = \{g_1\} = \{1\}$. Eq. (5) becomes

$$\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) \times 1 \, \in \mathcal{J}_{i_1\cdots i_n} \, , \qquad (9)$$

thus $\mathcal{I}_{i_1\cdots i_n}$ is reducible. $\square$

2.2. Integrand Recursion Formula

After substituting Eqs. (5) and (6) in Eq. (3), we get a non-homogeneous recurrence relation for the n-denominator integrand,

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{n} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}}. \qquad (10)$$

According to Eq. (10), $\mathcal{I}_{i_1\cdots i_n}$ is expressed in terms of integrands, $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n}$, with $(n-1)$ denominators. $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n}$ are obtained from $\mathcal{I}_{i_1\cdots i_n}$ by pinching the i_κ-th denominator. The numerator of the non-homogeneous term is the remainder $\Delta_{i_1\cdots i_n}$ of the division (5). By construction, it contains only irreducible monomials with respect to $\mathcal{G}_{i_1\cdots i_n}$, thus it is identified with the *residue* at the cut $(i_1 \ldots i_n)$.

The integrands $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}$ can be decomposed repeating the procedure described in Eqs. (3)-(5). In this case the polynomial division of $\mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}$ has to be performed modulo the Gröbner basis of the ideal $\mathcal{J}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}$, generated by the corresponding $(n-1)$ denominators.

The complete multi-pole decomposition of the integrand $\mathcal{I}_{i_1\cdots i_n}$ is achieved by successive iterations of Eqs. (3)-(5). Like an Erathostene's sieve, the recursive application of Eqs. (5) and (10) extracts the unique structures of the remainders Δ's. The procedure naturally stops when all cuts are exhausted, and no denominator is left, leaving us with the integrand reduction formula.

If all quotients of the last divisions vanish, the integrand is *cut-constructible*, i.e. it can be determined by sampling the numerator on the solutions of the cuts. If the quotients do not vanish, they give rise to *non-cut-constructible terms*, i.e. terms vanishing at every multi-pole. They can be reconstructed by sampling the numerator away from the cuts. Non-cut-constructible terms may occur in non-renormalizable theories, where the rank of the numerator is higher than the number of denominators [26].

The Proposition 2.2 and the recurrence relation (10) are the two mathematical properties underlying the integrand decomposition of any scattering amplitudes. The polynomial form of each residue is univocally derived from the division modulo the Gröbner basis of the corresponding cut.

3. One-loop integrand decomposition

In this section we decompose an n-point integrand $\mathcal{I}_{0\cdots(n-1)}$ of rank-n with $n > 5$, using the procedure described in Section 2. The reduction of higher-rank and/or lower-point integrands proceeds along the same lines.

In d-dimensions, the generic n-point one-loop integrand reads as follows:

$$\mathcal{I}_{0\cdots(n-1)} \equiv \frac{\mathcal{N}_{0\cdots(n-1)}(q,\mu^2)}{D_0(q,\mu^2)\cdots D_{n-1}(q,\mu^2)} \, . \qquad (11)$$

We closely follow the notation of [26, 35]. Objects living in $d = 4-2\epsilon$ are denoted by a bar, e.g. $\bar{\not{q}} = \not{q} + \not{\mu}$ and $\bar{q}^2 = q^2 - \mu^2$.

For later convenience, for each $\mathcal{I}_{i_1\cdots i_k}$ we define a basis $\mathcal{E}^{(i_1\cdots i_k)} = \{e_i\}_{i=1,\ldots,4}$.
If $k \geq 5$, then $e_i = k_i$, where k_i are four external momenta.
If $k < 5$, then e_i are chosen to fulfill the following relations:

$$\begin{aligned} &e_1^2 = e_2^2 = 0 \, , && e_1 \cdot e_2 = 1 \, , \\ &e_3^2 = e_4^2 = \delta_{k4} \, , && e_3 \cdot e_4 = -(1-\delta_{k4}) \, . \end{aligned} \qquad (12)$$

In terms of $\mathcal{E}^{(i_1\cdots i_k)}$, the loop momentum can be decomposed as,

$$q^\mu = -p_{i_1}^\mu + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu \, . \qquad (13)$$

Accordingly, each numerator $\mathcal{N}_{i_1\cdots i_k}$ can be treated as a rank-k polynomial in $\mathbf{z} \equiv (x_1, x_2, x_3, x_4, \mu^2)$,

$$\mathcal{N}_{i_1\cdots i_k} = \sum_{\vec{j}\in J(k)} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5} \, , \qquad (14)$$

with $J(k) \equiv \{\vec{j} = (j_1,\ldots,j_5) : j_1 + j_2 + j_3 + j_4 + 2\, j_5 \leq k\}$.

Step 1. When $n > 5$, the Proposition 2.2 guarantees that $\mathcal{N}_{0\cdots n-1}$ is reducible, and, by iteration, it can be written as a linear combination of 5-point integrands $\mathcal{I}_{i_1\cdots i_5}$.

Step 2. The numerator of each $\mathcal{I}_{i_1\cdots i_5}$ is a rank-5 polynomial in $\mathbf{z}$, cfr. Eq. (14). We define the ideal $\mathcal{J}_{i_1\cdots i_5}$, and compute the Gröbner basis $\mathcal{G}_{i_1\cdots i_5} =$

$(g_1, \ldots, g_5)$, which is found to have a remarkably simple form:

$$g_i(\mathbf{z}) = c_i + z_i \;, (i = 1, \ldots, 5) \;. \tag{15}$$

We observe that each g_i depends *linearly* on the i-th component of $\mathbf{z}$.

The division of $\mathcal{N}_{i_1\cdots i_5}$ modulo $\mathcal{G}_{i_1\cdots i_5}$, see Eq.(5), gives a *constant* remainder,

$$\Delta_{i_1\cdots i_5} = c_0 \;. \tag{16}$$

The term $\Gamma_{i_1\cdots i_5}$ in Eq. (6) is,

$$\Gamma_{i_1\cdots i_5} = \sum_{\kappa=1}^{5} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \;,$$

where $\mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}$ are the numerators of the 4-point integrands, $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}$, obtained by removing the i_κ-th denominator.

Step 3. For each $\mathcal{I}_{i_1\cdots i_4}$, the numerator $\mathcal{N}_{i_1\cdots i_4}$ is a rank-4 polynomial in $\mathbf{z}$. The Gröbner basis $\mathcal{G}_{i_1\cdots i_4}$ of the ideal $\mathcal{J}_{i_1\cdots i_4}$ contains four elements. Dividing $\mathcal{N}_{i_1\cdots i_4}$ modulo $\mathcal{G}_{i_1\cdots i_4}$, we obtain the remainder. The latter depends on μ^2 and on the fourth component of the loop momentum q in the basis $\mathcal{E}^{(i_1\cdots i_4)}$,

$$\begin{aligned} \Delta_{i_1\cdots i_4} &= c_0 + c_1 x_4 \\ &+ \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4) \,. \end{aligned} \tag{17}$$

The term $\Gamma_{i_1\cdots i_4}$,

$$\Gamma_{i_1\cdots i_4} = \sum_{\kappa=1}^{4} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \;,$$

contains the numerators of 3-point integrands $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}$.

Step 4. The Gröbner basis $\mathcal{G}_{i_1 i_2 i_3}$ is formed by three elements, and is used to divide $\mathcal{N}_{i_1 i_2 i_3}$. The remainder $\Delta_{i_1 i_2 i_3}$ is polynomial in μ^2 and in the third and fourth components of q in the basis $\mathcal{E}^{(i_1 i_2 i_3)}$,

$$\begin{aligned} \Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 \\ &+ c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 \\ &+ \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \;. \end{aligned} \tag{18}$$

The term $\Gamma_{i_1 i_2 i_3}$ generates the rank-2 numerators of the 2-point integrands $\mathcal{I}_{i_1 i_2}$, $\mathcal{I}_{i_1 i_3}$, and $\mathcal{I}_{i_2 i_3}$.

Step 5. The remainder of the division of $\mathcal{N}_{i_1 i_2}$ by the two elements of $\mathcal{G}_{i_1 i_2}$ is:

$$\begin{aligned} \Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 \\ &+ c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 \\ &+ c_9 x_2 x_4 + c_9 \mu^2 \;. \end{aligned} \tag{19}$$

It is polynomial in μ^2 and in the last three components of q in the basis $\mathcal{E}^{(i_1 i_2)}$. The reducible term of the division, $\Gamma_{i_1 i_2}$, generates the rank-1 integrands, $\mathcal{I}_{i_1}$, and $\mathcal{I}_{i_2}$.

Step 6. The numerator of the 1-point integrands is linear in the components of the loop momentum in the basis $\mathcal{E}^{(i_1)}$,

$$\mathcal{N}_{i_1} = \beta_0 + \sum_{j=1}^{4} \beta_j \, x_j \;.$$

The only element of the Gröbner basis $\mathcal{G}_{i_1}$ is D_{i_1}, which is quadratic in $\mathbf{z}$. Therefore the division modulo $\mathcal{G}_{i_1}$, leads to a vanishing quotient, hence

$$\mathcal{N}_{i_1} = \Delta_{i_1} \;. \tag{20}$$

Step 7. Collecting all the remainders computed in the previous steps, we obtain the complete decomposition of $\mathcal{I}_{0\cdots n-1}$ in terms of its multi-pole structure

$$\mathcal{I}_{0\cdots n-1} = \sum_{k=1}^{5} \left(\sum_{1=i_1<\ldots<i_k}^{n-1} \frac{\Delta_{i_1\cdots i_k}}{D_{i_1}\cdots D_{i_k}} \right) . \tag{21}$$

Eq. (21) reproduces the well-known one-loop d-dimensional integrand decomposition formula [5, 27–29, 35, 36].

We remark that the basis $\mathcal{E}^{(i_1\cdots i_k)}$, defined in Eq.(13) and used for decomposing the integration momentum q, depends only on the external momenta of diagram associate to the cut, eventually complemented by orthogonal elements. Therefore, $\mathcal{E}^{(i_1\cdots i_k)}$ can be used as well to decompose the integration momenta of multi-loop diagrams [14].

4. The Maximum-cut Theorem

At ℓ loops, in four dimensions, we define a *maximum-cut* as a (4ℓ)-ple cut

$$D_{i_1} = D_{i_2} = \cdots = D_{i_{4\ell}} = 0 \;, \tag{22}$$

which constrains completely the components of the loop momenta. In four dimensions this implies the presence of four constraints for each loop momenta. We assume that, in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one. Under this assumption we have the following

Theorem 4.1 (Maximum cut)**.** *The residue at the maximum-cut is a polynomial paramatrized by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

Proof. Let us parametrize the propagators using 4ℓ variables $\mathbf{z} = (z_1, \ldots, z_{4\ell})$. In this parametrization, the solutions of the maximum-cut read,

$$\mathbf{z}^{(i)} = \left(z_1^{(i)}, \ldots, z_{4\ell}^{(i)}\right) \ , \text{ with } i = 1, \ldots, n_s \ . \quad (23)$$

Let $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ be the ideal generated by the on-shell denominators, $\mathcal{J}_{i_1 \cdots i_{4\ell}} = \langle D_{i_1}, \ldots, D_{i_{4\ell}} \rangle$.
According to the assumptions, the number n_s of the solutions of (22) is finite, and each of them has multiplicity one, therefore $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ is zero-dimensional [22] and radical[1], see Cor. 2.6, Ch. 4 of [21]. In this case, the *Finiteness Theorem* (Prop. 8, Ch. 5 of [20]) ensures that the remainder of the division of any polynomial modulo $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ can be parametrized exactly by n_s coefficients.

Moreover, up to a linear coordinate change, we can assume that all the solutions of the system have distinct first coordinate z_1, i.e. $z_1^{(i)} \neq z_1^{(j)}$ $\forall\, i \neq j$. We observe that $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ and z_1 are in the *Shape Lemma* position (Prop. 2.3 of [24]) therefore a Gröbner basis for the lexicographic order $z_1 < z_2 < \cdots < z_n$ is $\mathcal{G}_{i_1 \cdots i_{4\ell}} = \{g_1, \ldots, g_{4\ell}\}$, in the form

$$\left\{ \begin{array}{lcl} g_1(\mathbf{z}) & = & f_1(z_1) \\ g_2(\mathbf{z}) & = & z_2 - f_2(z_1) \\ & \vdots & \\ g_{4\ell}(\mathbf{z}) & = & z_{4\ell} - f_{4\ell}(z_1) \ . \end{array} \right. \quad (24)$$

The functions f_i are univariate polynomials in z_1. In particular f_1 is a rank-n_s square-free polynomial [23],

$$f_1(z_1) = \prod_{i=1}^{n_s} \left(z_1 - z_1^{(i)}\right) \ , \quad (25)$$

i.e. it does not exhibits repeated roots. The multivariate division of $\mathcal{N}_{i_1 \cdots 1_{4\ell}}$ modulo $\mathcal{G}_{i_1 \cdots i_{4\ell}}$ leaves a remainder $\Delta_{i_1 \cdots i_{4\ell}}$ which is a univariate polynomial in z_1 of degree $(n_s - 1)$ [25], in accordance with the *Finiteness Theorem.* □

The maximum-cut theorem ensures that the maximum-cut residue, at any loop, is completely

[1] Given an ideal $\mathcal{J}$, the *radical* of $\mathcal{J}$ is $\sqrt{\mathcal{J}} \equiv \{f \in P[\mathbf{z}] : \exists\, s \in \mathbb{N},\ f^s \in \mathcal{J}\}$. $\mathcal{J}$ is radical iff $\mathcal{J} = \sqrt{\mathcal{J}}$.

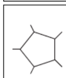
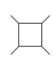
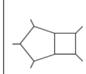
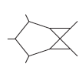
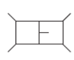
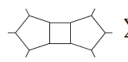
diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8

Figure 1: The on-shell diagrams in the picture are examples of maximum-cuts. The first diagram in the left column represents the 5-ple cut of the 5-point one-loop dimensionally regulated amplitude. All the other on-shell diagrams are considered in four dimensions. For each of them, the general structure of the residue Δ (according to the Shape Lemma) and the corresponding value of n_s are provided.

determined by the n_s distinct solutions of the cut-conditions. In particular it can be reconstructed by sampling the integrand on the solutions of the maximum cut itself.

At one loop and in $(4 - 2\epsilon)$-dimensions, the 5-ple cuts are maximum-cuts. The remarkably simple structure of the Gröbner basis in Eq. (15) is dictated by the maximum-cut theorem. Moreover in this case $n_s = 1$, thus the residue in Eq. (16) is a constant.

The structures of the residues of the maximum cut, together with the corresponding values of n_s, for a set of one-, two-, and three-loop diagrams are collected in Figure 1.

The calculations of Sections 3 and 4 have been carried out using the package S@M [37] and the functions `GroebnerBasis` and `PolynomialReduce` of Mathematica, respectively needed for the generation of the Gröbner basis and the polynomial division.

5. Conclusions

We presented a new algebraic approach, where the evaluation of scattering amplitudes is addressed by using multivariate polynomial division, with the components of the loop-momenta as indeterminates. We found a recurrence relation to construct the integrand decomposition of arbitrary scattering amplitudes, independently of the number of loops. The recursive algorithm is based on the Weak Nullstellensatz Theorem and on the division modulo the Gröbner basis associated to all possible multiparticle cuts. Using this technique, we rederived

the well-known one-loop integrand decomposition formula. Finally, by means of the Finiteness Theorem and of the Shape Lemma, we proved that the residue at the maximum-cuts is parametrised exactly by a number of coefficients equal to the number of solutions of the cut itself.

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